Infinite Free Resolutions Induced by Pommaret-like Bases (Extended Abstract)

Amir Hashemi[1,2,⋆], Matthias Orth[3], and Werner M. Seiler[3]

[1] Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran,
[2] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, 19395-5746, Iran, Amir.Hashemi@iut.ac.ir
[3] Institut für Mathematik, Universität Kassel, 34109 Kassel, Germany, {morth,seiler}@mathematik.uni-kassel.de

Abstract. Free resolutions are an important tool in algebraic geometry for the structural analysis of modules over polynomial rings and their quotient rings. Minimal free resolutions are unique up to isomorphism and induce homological invariants in the form of Betti numbers. It is known that Pommaret bases of ideals in the polynomial ring induce finite free resolutions and that the Castelnuovo-Mumford regularity and projective dimension can be easily obtained already from the Pommaret basis. In this article, we generalize this construction to Pommaret-like bases, which are generally smaller. We apply Pommaret-like bases also to infinite resolutions over quotient rings. Over Clements-Lindström rings, we derive module bases for resolution using only the Pommaret-like basis. Finally, restricting to monomial ideals over the polynomial ring, we derive an explicit formula for the differential of the induced resolution.

1 Infinite Free Resolutions

We work over the quotient rings of a polynomial ring $\mathcal{R} = K[x_1, \dots, x_n]$ over a field K. Thus given a homogeneous ideal $\mathcal{I} \subseteq \mathcal{R}$, we make the computations in the ring $\mathcal{R}/\mathcal{I}$. A special case for $\mathcal{I} = \{0\}$ is the ring $\mathcal{R}$ itself. We write $\mathcal{T}$ for the set of all terms $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \in \mathcal{R}$ with $\mu = (\mu_1, \dots, \mu_n)$.

We construct free resolutions of homogeneous ideals $\mathcal{J} \trianglelefteq \mathcal{R}/\mathcal{I}$. A free resolution $\mathbf{F}$ of $\mathcal{J}$ is given by finitely generated free $\mathcal{R}/\mathcal{I}$-modules $F_0, F_1, \dots$ and homogeneous $\mathcal{R}/\mathcal{I}$-linear maps $\delta_0, \delta_1, \delta_2, \dots$ as in the following diagram:

$$\mathbf{F}: \quad \cdots \xrightarrow{\delta_{m+2}} F_{m+1} \xrightarrow{\delta_{m+1}} F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{J} \to 0,$$

with $\mathrm{im}(\delta_0) = \mathcal{J}$ and $\mathrm{im}(\delta_{m+1}) = \ker(\delta_m)$ for all $m \geq 0$. The collection $\{\delta_m\}_{m \geq 0}$ of maps is called the *differential* of the resolution. Leaving aside degree shifts, we can write $F_m = (\mathcal{R}/\mathcal{I})^{r_m}$ for $m \geq 0$. Each δ_m is described by the images $\delta(\mathbf{e}_i)$, $i \in \{1, \dots, r_m\}$; equivalently, δ_m is represented by a matrix $D_m \in (\mathcal{R}/\mathcal{I})^{r_{m-1} \times r_m}$, whose i-th column is $\delta_m(\mathbf{e}_i)$. Moreover, $D_m \cdot D_{m+1} = 0$ for all m.

⋆ A. Hashemi's research was in part supported by a grant from IPM (No. 1401130214).

The set $G := \{\delta_0(\mathbf{e}_1), \dots, \delta_0(\mathbf{e}_{r_0})\}$ is a homogeneous generating set of $\mathcal{J}$ and the columns of D_1 form a homogeneous generating set G_1 of $\mathrm{Syz}(G)$. Generally, the set G_m of columns of D_m is a homogeneous generating set of the iterated syzygy module $\mathrm{Syz}^m(G)$. For a more detailed introduction to infinite graded free resolutions and an overview of associated research questions, see [7].

An open-ended question is how to find explicit formulas for the differentials of monomial resolutions. Our work generalizes some related constructions and expands the class of monomial ideals with known explicit differential formulas.

2 Pommaret-like Bases

Pommaret-like bases are an adapted form of Pommaret bases, which are types of involutive bases. Involutive bases are Gröbner bases with special combinatorial properties introduced by Gerdt and Blinkov [4]. An overview of theory, algorithms and applications can be found in [9]. They arise via a restriction of the usual divisibility relation of terms to an involutive division.

We shortly describe the Pommaret division. The *class* of a term $1 \neq x^\mu \in \mathcal{T}$ with $\mu = (\mu_1, \dots, \mu_n)$ is defined as the index $\mathrm{cls}\,(x^\mu) = \min\{i \mid \mu_i \neq 0\}$. A variable x_i is Pommaret multiplicative for x^μ, if $i \leq \mathrm{cls}(x^\mu)$. Not every monomial ideal has a finite Pommaret basis; those that do are called *quasi-stable.* A polynomial ideal $\mathcal{I}$ is in *quasi-stable position*, if its leading ideal with respect to a term-order is quasi-stable. A monomial ideal $\mathcal{I}$ is called *stable*, if its minimal basis is a Pommaret basis.

The Pommaret-like division is based on the Janet-like division introduced by Gerdt and Blinkov [5]. Let $U \subset \mathcal{T}$ be a finite set of terms. For any term $u \in U$ and any index $1 \leq i \leq n$, a *non-multiplicative power* of u for the *Janet-like division* at the variable x_i exists, if there is a term $v \in U$ with $\deg_j(v) = \deg_j(u)$ for all $i < j \leq n$ and $\deg_i(v) > \deg_i(u)$. The non-multiplicative power is then given by $x_i^{h_i(u,U)}$, where $h_i(u,U)$ is the minimal positive value of $\deg_i(v) - \deg_i(u)$ as v ranges over all terms in U with the properties just mentioned. The set of all non-multiplicative powers of $u \in U$ is denoted by $\mathrm{NMP}(u,U)$.

Definition 1. *The* Pommaret-like division P *assigns to each term $t \in \mathcal{T}$ contained in a finite set of terms $U \subset \mathcal{T}$, the following non-multiplicative powers:*

- *All Janet-like non-multiplicative powers $x_a^{p_a}$ with $a > \mathrm{cls}(t)$.*
- *The variables x_b with $b > \mathrm{cls}(t)$ for which there exists no Janet-like non-multiplicative power.*

For a given monomial ideal, a finite Pommaret-like basis exists if and only if the ideal is quasi-stable. This basis is general smaller than the Pommaret basis of the same ideal; for some types of ideals like toric ones much smaller.

We extend the Pommaret-like division to ideals in monomial quotient rings. For a quasi-stable ideal $\mathcal{I}$ and a monomial ideal $\mathcal{J} \subseteq \mathcal{R}/\mathcal{I}$, a Pommaret-like basis for $\mathcal{J}$ exists if and only if $\mathcal{J}$ is the image of a quasi-stable ideal in $\mathcal{R}$. We write $P_{\mathcal{I}}$ for the Pommaret-like division in $\mathcal{R}/\mathcal{I}$. To emphasize that we are working in a quotient ring, we speak of a *relative Pommaret-like basis.*

3 Applications to Resolutions

In [6], the syzygies of relative Pommaret bases were analysed, using a suitable module term ordering and applying a construction due to Schreyer [8]. We generalize this to relative Pommaret-like bases in quotient rings $\mathcal{R}/\mathcal{I}$ defined by an irreducible quasi-stable monomial ideal $\mathcal{I}$. A Pommaret-like basis of the leading module terms of the syzygy module of a relative Pommaret basis is given by the non-multiplicative powers together with the annihilating factors modulo $\mathcal{I}$ not divisible by such a non-multiplicative power. By iteration, we obtain a free resolution together with Pommaret-like bases for the first and all higher syzygy modules. These Pommaret-like bases are reduced Gröbner bases.

Theorem 2. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a quasi-stable monomial ideal generated by the minimal Pommaret-like basis $H \subset \mathcal{I} \cap \mathcal{T}$. Assume that H is also the minimal monomial generating set of $\mathcal{I}$ and that for each non-multiplicative power $x_j^{p_j}$ of $t \in H$, the term $(t/x_{\mathrm{cls}(t)})x_j^{p_j}$ has a Pommaret-like divisor in H. Then the free resolution of $\mathcal{I}$ induced by the basis H is the minimal free resolution of $\mathcal{I}$ over $\mathcal{R}$.*

Adapting the definitions due to [1] to our conventions on variable orderings, we say that an irreducible, non-zero monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is *Clements-Lindström*, if its minimal generating set is of the form $\{x_i^{a_i}, x_{i+1}^{a_{i+1}}, \ldots, x_n^{a_n}\}$ with $2 \leq a_n \leq a_{n-1} \leq \cdots \leq a_{i+1} \leq a_i$. We call $\mathcal{R}/\mathcal{I}$ a *Clements-Lindström ring.*

Theorem 3. *Let $\mathcal{I} \trianglelefteq \mathcal{R}$ be a Clements-Lindström ideal and let $\mathcal{J} \supset \mathcal{I}$ be a monomial ideal generated by the minimal Pommaret-like basis $H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$ relative to $\mathcal{I}$. Assume that H is also the minimal monomial generating set of $\mathcal{J}$ relative to $\mathcal{I}$ and that for each $t \in H$ and $x_a^{p_a} \in \mathrm{NMP}_{P_\mathcal{I}}(t, H)$, the unique $P_\mathcal{I}$-divisor $s \in H$ of $t \cdot x_a^{p_a}$ fulfils $\mathrm{cls}(s) > \mathrm{cls}(t)$. Then the free resolution of $\mathcal{J}$ over $\mathcal{R}/\mathcal{I}$ induced by the basis H is the minimal free resolution of $\mathcal{J}$ over $\mathcal{R}/\mathcal{I}$.*

Consider a Pommaret-like basis H relative to $\mathcal{I} = \langle x_k^{h_k}, x_{k+1}^{h_{k+1}} \ldots, x_n^{h_n} \rangle$. We write $\mathrm{supp}(\mathcal{I}) := \{x_k, x_{k+1}, \ldots, x_n\}$. The induced free resolution is supported on free $\mathcal{R}$-modules. The first free $\mathcal{R}$-module M_0 has a basis that we enumerate as $\{\mathbf{e}_\alpha \mid h_\alpha \in H\}$.

The ideals $\mathcal{J}_\alpha = \langle x_i^{d_i} \mid x_i^{d_i} \cdot \mathbf{e}_\alpha \in \mathrm{lt}(\mathrm{Syz}(H)) \rangle$ are irreducible and we write $\mathrm{supp}(\mathcal{J}_\alpha)$ for the set of variables appearing in their respective generating sets.

For the r-th module F_r in the resolution, we obtain a basis made of elements of the form $\mathbf{e}_{\alpha, x^\mu}$, where x^μ is a term of degree r with $\mathrm{cls}(x^\mu) \geq \mathrm{cls}(t_\alpha)$. Moreover, x^μ is supported on $\mathrm{supp}(\mathcal{J}_\alpha)$, and its projection onto $\mathrm{supp}(\mathcal{J}_\alpha) \setminus \mathrm{supp}(\mathcal{I})$ is square-free.

We apply our results to square-free Borel ideals over zero-dimensional Clements-Lindström rings and thus show that the minimal free resolution for such ideals found in [3] is a Pommaret-like induced resolution.

In the last part of our work, we restrict our attention to monomial ideals in the polynomial ring $\mathcal{R}$. For some classes of quasi-stable ideals we derive an explicit formula for the differential of the Pommaret-like induced resolution.

Definition 4. *Let $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the quasi-stable ideal $\mathcal{I} = \langle H \rangle$. For each $h_\alpha \in H$, and for each of its non-multiplicative powers $x_a^{p_a}$, there exists exactly one Pommaret-like divisor $h_\beta \in H$. We write $\Delta(\alpha, a) = \beta$, and $t_{\alpha,a} = (x_a^{p_a} \cdot h_\alpha)/h_\beta$ for the multiplicative cofactor. The ideal $\mathcal{I}$ is called* Δ-commuting*, if Δ satisfies the following properties:*

1. *If $b > a > \operatorname{cls}(h_\alpha)$ are two non-multiplicative indices and $\operatorname{cls}(h_{\Delta(\alpha,b)}) < a$, then the exponent of the non-multiplicative power of $h_{\Delta(\alpha,b)}$ at the variable x_a equals that of the non-multiplicative power of h_α at the variable x_a.*
2. *We have $\Delta(\Delta(\alpha, a), b) = \Delta(\Delta(\alpha, b), a)$.*
3. *We have $t_{\alpha,a} \cdot t_{\Delta(\alpha,a),b} = t_{\alpha,b} \cdot t_{\Delta(\alpha,b),a}$.*

Theorem 5. *Let $H = \{h_\alpha \mid \alpha \in A\} \subset \mathcal{T}$ be the minimal Pommaret-like basis of the Δ-commuting quasi-stable ideal $\mathcal{I} = \langle H \rangle$. The Pommaret-like induced resolution of $\mathcal{I}$ is supported on free generators of the form $\mathbf{e}_{\alpha,x^\mu}$, where the x^μ are square-free terms supported on $\{x_j \mid j > \operatorname{cls}(h_\alpha)\}$. The differential δ of the resolution is given by $\delta(\mathbf{e}_{\alpha,1}) = h_\alpha$, and, for $\deg(x^\mu) > 0$,*

$$\delta(\mathbf{e}_{\alpha,x^\mu}) = \sum_{x_j \in \operatorname{supp}(x^\mu)} (-1)^{\operatorname{sgn}(x_j, \operatorname{supp}(x^\mu))} \cdot \left(x_j^{p_j} \mathbf{e}_{\alpha, x^\mu/x_j} - t_{\alpha,j} \mathbf{e}_{\Delta(\alpha,j), x^\mu/x_j} \right). \quad (1)$$

In this formula, the summands which involve a non-existent free generator $\mathbf{e}_{\beta,x^\nu}$, i.e., one for which $\operatorname{supp}(x^\nu) \not\subseteq \{x_j \mid j > \operatorname{cls}(h_\beta)\}$, are zero by definition.

Theorem 5 generalizes the well-known resolution formula for stable monomial ideals due to Eliahou and Kervaire [2], as well as the resolution formula for quasi-stable monomial ideals due to Seiler [9, Thm. 5.4.18].

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