A Mathematica Package for Construction and Inversion of Analytic Mappings with Unit Jacobian

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Abstract. The set of polynomial mappings from the n-dimensional complex space into itself whose Jacobian matrix has nonzero constant determinant is known to be very vast in any dimension that exceeds one. The celebrated Jacobian Conjecture states that any such mapping is polynomially invertible. While computation of the determinant of the Jacobian matrix is very well supported in modern computer algebra systems, algorithmic inversion of a polynomial mapping is still a task of formidable computational complexity.
We will present a `Mathematica` package `JC` that can be used for construction and inversion of polynomial and certain more general analytic mappings with unit determinant of the Jacobian matrix. The package comprises functions that allow one to algorithmically construct such mappings for a given dimension of the space of variables and a given degree of the components of the mapping. We furthermore will discuss tools for computing the inverses of analytic mappings with unit determinant of the Jacobian matrix and compare their efficiency with that of existing solutions. The package together with a library of datasets used in and obtained by means of computational experiments are available for free public use at `https://www.researchgate.net/publication/358409332_JC_Package_and_Datasets`.

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Let $f = (f_1, \ldots, f_n)$ be an n-tuple of analytic functions in n complex variables $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ defined in a nonempty domain $D \subset \mathbb{C}^n$. We will call f a *Jacobian mapping* if the determinant of its Jacobian matrix is a nonzero complex number:

$$J(f; x) = J(f_1, \ldots, f_n; x_1, \ldots, x_n) := \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}. \qquad (1)$$

Here and throughout the text we will call this determinant *the Jacobian of the mapping* f as long as this does not lead to ambiguity.

By the inverse mapping to f we mean the analytic mapping $f^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{Id}$ in a nonempty domain in $\mathbb{C}^n$. The domain where these equalities are valid is in general heavily dependent on the mapping f and on the domain D. We will not investigate this dependence since all mappings under study are either defined by entire functions or are analytically extendible into the whole of the n-dimensional complex space with a possible exception for certain singular hypersurface $\mathcal{H}$. Such a mapping (which is in general multi-valued) is uniquely defined by any of its germs at a nonsingular point which can be further analytically extended into $\mathbb{C}^n \setminus \mathcal{H}$.

If $n = 1$, then any mapping satisfying (1) is of the form $ax + b$ with $a \in \mathbb{C}^*$. The inverse mapping is also affine linear. From now on, we will disregard this trivial case and assume that $n \geq 2$.

The famous Jacobian Conjecture [6] states that a mapping $f = (f_1, \ldots, f_n)$ defined by polynomials $f_j \in \mathbb{C}[x_1, \ldots, x_n]$ is Jacobian if and only if its inverse is a polynomial mapping, too. In any dimension $n \geq 2$ the family of all polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^n$ with unit Jacobian is extremely vast and, judging by known results, has a highly complex structure (see [2, Chapters 3,5], [7] and references therein). Constructing polynomial automorphisms of a given space is an important direction of research pursued in numerous publications [3–5, 10]. The seminal theorem by Drużkowski [1] ensures that in order to verify the Jacobian Conjecture it suffices to check its veracity for the very special class of cubic polynomial mappings of the form

$$\{(f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n, \quad f_j(x) = x_j + \left(\sum_{k=1}^{n} a_{jk} x_k\right)^3, \ j = 1, \ldots, n\}.$$

Here (a_{jk}) is a square matrix of size n with complex entries.

Motivated by the Drużkowski theorem, we investigate a wider family of analytic mappings defined by a square matrix $A = (a_{jk})$ of size $n \geq 2$ together with a function $\varphi(\zeta) \in \mathcal{O}(\Omega)$ that is analytic in a nonempty domain $\Omega \subset \mathbb{C}$. This family comprises mappings of the form

$$f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n, \quad f[A, \varphi](x) := x + \varphi(Ax) \tag{2}$$

with the coordinates

$$f_j : x \mapsto x_j + \varphi\left(\sum_{k=1}^{n} a_{jk} x_k\right), \quad j = 1, \ldots, n$$

whose Jacobian is identically equal to a nonzero constant for any x such that all of f_j are well-defined. Throughout the text we will say that the matrix A together with an analytic function $\varphi(\zeta)$ as above form a *good pair*. Finding good pairs, investigating their properties, and inverting the corresponding mapping (2) is the fundamental task that the `JC` package has been designed and optimized for.

Let U be a square matrix such that the Jacobian of the mapping $f[U, \varphi](x)$ is a nonzero constant for any x in its domain of definition and moreover for any analytic function $\varphi \in \mathcal{O}(\Omega)$. We will call such matrices *universal.* The following theorem holds.

Theorem 1. *(See [8, Theorem 4.13]) A universal matrix of size n is defined by an integer partition $p = (p_1, \ldots, p_m)$ of n and a permutation on m elements uniquely up to a choice of permutations similarities and values of algebraically independent complex parameters.*

We define the action of a univariate analytic function $\varphi(\cdot) \in \mathcal{O}(\Omega)$ on a complex vector $\xi = (\xi_1, \ldots, \xi_n) \in \Omega \times \ldots \times \Omega \subset \mathbb{C}^n$ to be termwise: $\varphi(\xi_1, \ldots, \xi_n) := (\varphi(\xi_1), \ldots, \varphi(\xi_n))$, unless otherwise is explicitly stated. For any $d = 2, 3, \ldots$ there exists an n-parametric family of square matrices $H(s), s \in \mathbb{C}^n$ such that for any universal matrix U the mapping $x + ((U \odot H(s))x)^d$ defined by the Hadamard product $U \odot H(s)$ has unit Jacobian. In [8] any such mapping has been proved to be polynomially invertible and an explicit recursive formula for its inverse has been given. We will discuss the computational frontiers of the class of Jacobian mappings that can be efficiently investigated by means of the algorithms implemented in the `JC` package.

All of the polynomial mappings to be presented are polynomially invertible. Moreover, for a universal matrix U and any analytic function $\varphi(\zeta)$ the inverse to the mapping $f[U, \varphi]$ is a finite superposition of $\varphi(\zeta)$ and elementary arithmetic operations acting on the variables $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ (cf [9]). However, this does not hold for arbitrary matrix A and arbitrary analytic function $\varphi(\zeta)$ such that the Jacobian of the mapping $x + \varphi(Ax)$ identically equals 1. We employ the notion of Toeplitz matrices to construct examples of Jacobian mappings of the form $f[A, \log](x) := x + \log(Ax)$ and conclude that for a certain choice of a Toeplitz matrix A the inverse mapping $f[A, \log]^{-1}$ is not a finite superposition of the logarithmic function and elementary arithmetic operations.

The present research is a technical spin-off of a project aimed at constructive description and algorithmic inversion of multivariate analytic mappings with unit Jacobian. The theoretical part of the conducted research is exposed in [8].

The `Mathematica` package `JC` comprising functions for construction, parameterization, inversion, and investigation of analytic mappings with unit Jacobian together with a library of datasets used in and obtained by means of computational experiments are available for free public use at `https://www.researchgate.net/publication/358409332_JC_Package_and_Datasets`.

The package is organized into three main blocks of functions. The first block comprises linear algebra functions that allow one to detect permutation similarity of square matrices with numeric and symbolic entries, compute principal minors of a square matrix and their sums, to manipulate with circulant and Vandermonde-type matrices etc.

The second block of functions in the package aims at algorithmic construction of polynomial and analytic mappings with unit Jacobian. The central function in this part allows one to compute the general universal matrix in a given dimension defined by given integer partition of the dimension of the space of variables and a permutation of a set of integers. Besides, this part of the package contains functions that allow one to compute Jacobian equations in given dimension and degree, compute the subsystem of simple Jacobian equations, as well as the equations that are satisfied if and only if the corresponding matrix forms a good pair with the logarithmic function.

Finally, the third group of functions in the package is designed to invert polynomial and analytic mappings of the form $f(x) = x + \varphi(Ax)$ with unit Jacobian defined by a square matrix A of size n and a univariate analytic function $\varphi(\zeta)$.

The routines of the `JC` package allow one to find matrices that form good pairs with analytic functions other than monomials or generic holomorphic functions. For instance, using the command `JEquationsLOG[A]` we obtain a system of 35 homogeneous algebraic equations in the variables a_{jk} that is satisfied if and only if the mapping $x + \log(Ax)$ has unit Jacobian. One of the families of solutions to this system of equations is given by the nilpotent matrix

$$\begin{pmatrix} 1 & a_{12} & a_{13} & -1 \\ a_{21} & 1 & -1 & -a_{21} \\ a_{21} & 1 & -1 & -a_{21} \\ 1 & a_{12} & a_{13} & -1 \end{pmatrix}$$

with a_{12}, a_{13}, and a_{21} assuming arbitrary complex values.

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