Implementation report on computing Gröbner bases over exterior algebra (extended abstract)*

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1 Introduction

A *Gröbner basis* is defined as a special kind of generator set for an ideal, and it gives a computational tool to determine many properties of the ideal. The notion of Gröbner bases has been extended to various noncommutative algebras such as free non-commutative algebra [2], [9] and exterior algebra [1]. In particular, a Gröbner basis in an exterior algebra $E = \bigwedge V$ is defined as a finite generator set for its two-sided ideal, where V is a K-vector space of dimension N with a fixed basis $\{e_1, \ldots, e_N\}$. One of the most different points (in a computational point of view) from the polynomial ring case is the existence of zero-divisors; more strongly the multiplication in E is *alternating*, i.e., $f \wedge f = 0$ for all $f \in V$. A typical application of the computation of Gröbner bases in exterior algebra is the efficient computation of the cohomology groups of coherent sheaves (on a projective space). More precisely, it is shown in [5] that the computation of the sheaf cohomology is reduced to that of a graded free resolution over an exterior algebra via the BGG correspondence [3]. Since such a free resolution is constructed by computing Gröbner bases for graded modules over an exterior algebra, the efficient Gröbner basis computation in E is required. As for algorithms to compute Gröbner bases in E, modified Buchberger's algorithm and Faugére's F_4 can be constructed, and they have been implemented in Magma, Singular, and Macaulay2. However, contrary to the development of implementations, we could not find any analysis on these algorithms, in theory nor in practice yet.

In this extended abstract, we report practical behaviors of the Gröbner basis computation over exterior algebra, with some important remarks for accurate and efficient implementations. As preliminary experimental results obtained by our current implementations, we also show the (Macaulay) matrix size and the maximum total-degree appearing in the execution of the F_4 algorithm.

2 Preliminaries

This section briefly reviews the definition of an exterior algebra with its some properties, and the computation scheme of Gröbner bases.

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Let K be a field, and V a K-vector space of dimension N with a fixed basis $\{e_1, \dots, e_N\}$. We denote by $T(V)$ the tensor algebra, say $T(V) := \bigoplus_{i\geq 0} V^{\otimes i}$. The *exterior algebra* of V is defined as a graded K-algebra $\bigwedge V := T(V)/J(V)$, where $J(V)$ is the two-sided ideal generated by $\{f \otimes f : f \in V\} \subset T(V)$. For each element $f_1 \otimes \cdots \otimes f_k \in T(V)$, we denote by $f_1 \wedge \cdots \wedge f_k$ its image in $\bigwedge V$ by the canonical homomorphism $T(V) \to \bigwedge V$. The multiplication in $\bigwedge V$ is *alternating* (and thus *antisymmetric*) on elements in V, i.e., $f \wedge f = 0$ for all $f \in V$, and hence $f \wedge g = -g \wedge f$ for all $f, g \in V$. To simplify the notation, we denote $fg = f \wedge g$. For each d, the degree d homogeneous part of $\bigwedge V$ is $\bigwedge^d V := T(V)_d/J_d(V)$, where $T(V)_d = V^{\otimes d}$ and $J_d(V) = T(V)_d \cap J(V)$. We denote by $\bigwedge^k V$ the K-vector subspace generated by $\{f_1 \cdots f_k : f_i \in V\} \subset \bigwedge V$. It has a basis $\{e_{i_1} \cdots e_{i_k} : 0 \leq i_1 < \cdots < i_k \leq N\}$, and so $\dim_K \bigwedge^k V = \binom{N}{k}$. By $\bigwedge^k V = 0$ for $k > N$, we have $\bigwedge V = \bigoplus_{k=0}^N \bigwedge^k V$ and $\dim_K \bigwedge V = \sum_{k=0}^N \binom{N}{k} = 2^N$.

We here recall the necessary computational notion of a Gröbner basis in exterior algebra briefly. Elements of the form $e_{\underline{i}} = e_{i_1} \cdots e_{i_k}$ with $\underline{i} = (i_1, \dots, i_k)$ and the unity 1_E are called monomials of E, and the set of all monomials in E is denoted by $\mathcal{M}(E)$. For $e_{\underline{i}} \in \mathcal{M}(E)$, we set $t(e_{\underline{i}}) = x_{\underline{i}} := x_{i_1} \cdots x_{i_k}$ in the polynomial ring $S = K[x_1, \dots, x_N]$. A monomial order on E is defined by using a given monomial order $\succ$ on S, as follows: For $u, v \in \mathcal{M}(E)$, we say $u \succ v$ if $t(u) \succ t(v)$. The notation $\mathrm{LT}_\prec$, $\mathrm{LM}_\prec$ and $\mathrm{LC}_\prec$ are used for the leading term, the leading monomial, and the leading coefficient, respectively. For a subset $F \subset E$, we set $\mathrm{LT}_\prec(F) := \{\mathrm{LT}_\prec(f) : f \in F\}$ and $\mathrm{LM}_\prec(F) := \{\mathrm{LM}_\prec(f) : f \in F\}$. A *Gröbner basis* for a two-sided ideal $I \subset E$ with respect to $\prec$ is defined as a finite subset $G \subset E$ generating I with $\langle \mathrm{LT}_\prec(G)\rangle_E = \langle \mathrm{LT}_\prec(I)\rangle_E$.

As in the polynomial case, S polynomials in E can be defined, but an additional notion to construct Buchberger's criterion is the so-called T *polynomial*, whose definiton is as follows: Let $g \in E$. For each e_j with $1 \leq j \leq N$ such that $e_j\mathrm{LT}(g) = 0$, we call $e_j g$ a T polynomial of g. With T polynomials, Buchberger's criterion for E says that $G = \{g_1, \dots, g_m\} \subset E$ is a Gröbner basis if and only if all the S and T polynomials constructed from elements in G are reduced into 0 by G. Based on this criterion, a Bughberger algorithm is constructed in [1], and we call it the *modified Buchberger algorithm* in this extended abstract.

3 Our contribution

Here we give a list of our main issues aiming for efficient implementations:

- **Easy criteria:** For example, if an ideal I of E contains an element f with non-zero constant term, then the Gröbner basis of I is $\{1\}$.
- **Accurate implementation of the modified Buchberger algorithm:** In some non-homogeneous cases, the modified Buchberger algorithm is *not* directly applicable. For example, when $E = \bigwedge V$ for a $\mathbb{Q}$-vector space V with basis $\{e_1, e_2, e_3, e_4\}$, the modified Buchberger algorithm for the input $I = \{e_1e_2 + e_3\}$ outputs $\{e_1e_2 + e_3, e_1e_3, e_2e_3\}$. However, e_3e_4 is included in I and e_3e_4 is not reducible by the output, where e_3e_4 is calculated as

$((e_1e_2+e_3)e_4-e_4(e_1e_2+e_3))/2 = e_3e_4$. This means that the output is not a Gröbner basis. We have attempted this calculation in Magma and have the output, and this fact should be noted. To resolve this problem, it suffices to add e_if and fe_i with $f = e_1e_2 + e_3$ for all $1 \leq i \leq 4$ to the input, by which we can compute a correct Gröbner basis for $\langle f \rangle_E$. Note that such a case does not happen in the homogeneous case.

- **Selection strategy of S and T polynomials:** Consider the homogeneous case. When a Gröbner basis is calculated by the modified Buchberger algorithm, a selection strategy is important. Unlike polynomial rings, we need to process not only S polynomials but also T polynomials in the case of exterior algebra, and thus the efficiency of the algorithm can be affected by whether S or T polynomials are preferentially generated. The term order is also important for the total efficiency, which also needs to be considered. We shall report some efficient strategies, in theory or in heuristic, at the conference.
- **Application of the F_4 algorithm:** We made a preliminary implementation of the F_4 algorithm for the exterior algebra. To implement the F_4 algorithm correctly as in [6] for the polynomial case, we have some remarks on the treatment of T-polynomials: In the process of collecting reducers, the leading monomials of T polynomials should not be stored as monomials which can be reduced by reducers already collected (otherwise, one might not obtain a Gröbner basis).

4 Preliminary experimental results and remarks

Here, we report briefly the computational behavior of our current implementation which is important for examining the real Gröbner basis computation. For the computational efficiency, the (Macaulay) matrix size and the maximum total-degree are very crucial. Note that the maximum total-degree in the Gröbner basis computation is related to the *degree of regularity* for polynomial case.

Here we explain details of our experiments: In our experiments, we adopt our F_4 implementation where the degree reverse lexicographic order is used. For each set of parameters N, m and D, each input consists of m random dense homogeneous elements in E with N variables of the same degree D. We measured the maximum size of Macaulay matrices and the maximum total-degree appearing in the execution of the F_4 algorithm, and summarize them in Table 1 and Table 2 respectively.

From our experiments described as above, we observe the following:

Our current observation: For the exterior algebra, Table 1 implies that the matrix size might not be so large for larger D, compared with the polynomial case. On the other hand, we are guessing from Table 2 that the maximum total-degree of elements in E appearing in the Gröbner basis computation also becomes larger when D increases, as in the polynomial case. This is because the number of monomials in E of higher degree is smaller than that in a polynomial ring with the same number of variables. For example, when $N = 10$, the number of monomials in E with degree $d = 5$ is 252, which is the largest among the cases

$0 \leq d \leq 10$, while the number of monomials with degree 10 is only one. Therefore, even if the degree D of the input elements is higher, the computational complexity not always becomes large. Further research is needed to evaluate in more details the complexity of computing a Gröbner basis in exterior algebra. We are guessing the gap of the degree of the input polynomials and the maximum total-degree is very small, and at the conference, we address our theoretical observation.

Table 1. The maximum size of Macaulay matrices appearing in the execution of our F_4 implementation over E with $N = 10$ variables in the left and the theoretical size in the right, the number m and the degree D of each input set of elements in E.

		D		
		3	5	7
m	1	452×301 / 967×968	278×165 / 637×638	83×55 / 175×176
	3	656×427 / 2901×968	222×330 / 1911×638	265×55 / 525×176
	5	1148×365 / 4835×968	1376×134 / 3185×638	442×55 / 875×176

Table 2. The maximum total-degree of elements in E appearing in the execution of our F_4 implementation over E with $N = 10$ variables, the number m and the degree D of each input set of elements in E.

		D					
		2	3	4	5	6	7
m	1	6	7	7	7	8	9
	3	5	6	6	7	8	9
	5	4	5	6	7	8	8
	14	3	5	6	7	7	8

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