Computing models of orbifold del Pezzo surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ format ⋆

Muhammad Imran Qureshi[0000−0002−5268−7719]

Department of Mathematics, King Fahd University of Petroleum and Minerals, Saudi Arabia imran.qureshi@kfupm.edu.sa

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1 Introduction

1.1 Background

A classification of algebraic varieties is one of the fundamental questions in algebraic geometry. In particular, there are finite number of deformation families of Fano manifolds in each dimension. Fano varieties are defined by a positivity condition on the curvature, making them higher dimensional generalisations of the sphere: they include projective homogeneous spaces that are amongst the most commonly occurring varieties in applications to other subjects. Two dimensional Fano varieties are called del Pezzo surfaces.

Formally, an algebraic surface X is a called a *del Pezzo surface* if the anticanonical class $-K_X$ is ample. The *Fano index* I of X is the largest positive integer I such that $-K_X = I \cdot D$ for some divisor D in the class group of X. An orbifold point of type $\frac{1}{r}(a, b)$ is the quotient $\pi : \mathbb{A}^2 \to \mathbb{A}^2/\mu_r$ given by

$$\mu_r \ni \epsilon : (x_1, x_2) \mapsto (\epsilon^a x_1, \epsilon^b x_2).$$

It is called an isolated orbifold point if r is relatively prime to a and b.

Definition 1.1. [8] A *biregular model of orbifold del Pezzo surfaces* parametrized by positive integers is an infinite series of del Pezzo surfaces satisfying below conditions.

(i) A family of orbifold del Pezzo surfaces exists for each parameter $r(n)$ for all positive integers n.
(ii) The embedding $\mathbb{P}(w_i)$ of each family of surfaces has at least one weight equal to r and The rest of the weights w_i and $(-K_X)^2$ are functions of r.

If the affine cone over of the surface is smooth outside the origin then we call it to be quasismooth. A wellformed surface does not contain any orbifold singularities in codimension 1. A rigid del Pezzo surface is a del Pezzo surface who's singularities are not smoothable by $\mathbb{Q}$-Gorenstein deformations [5].

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Definition 1.2. [1] Let $Q = \frac{1}{r}(a,b)$ be an orbifold point and $d = \gcd(a+b, r)$, $k = (a+b)/d$ and $m = r/d$. Then we can present Q as $\frac{1}{dm}(1, mk-1)$ and if $d < m$ then it is called a (rigid) R-singularity.

One of the interesting problems is to construct the algebraic varieties embedded in some weighted projective space that can be described in terms of explicit equations. This allows us to use the properties of their graded rings to study their various geometric properties.

We construct two types of biregular models of rigid odPs which we call *biregular weight models* and *biregular index models*. A weight model consists of wellformed and quasismooth rigid odPs of a fixed Fano index and varying sets of weights of the ambient $\mathbb{P}^6(w_i)$ and in an index model the Fano index and set of weights of $\mathbb{P}^6(w_i)$ both vary for each family of del Pezzo surfaces. A series of hypersurface $X_{2r} \hookrightarrow \mathbb{P}(1,1,r,r)$ of Fano index 2 with two singular points of type $\frac{1}{r}(1,1)$ for $r = n+4, n \in \mathbb{Z}_+$ is an example of a rigid weight model. A simplest example of a rigid index model is the series $\mathbb{P}(1,1,r)$ of index $r+2$ having an orbifold point of type $\frac{1}{r}(1,1)$ for each $r = n+4, n \in \mathbb{Z}_+$.

1.2 Results

We construct biregular models of *orbifold rigid del Pezzo* surfaces of codimension 4 whose equations can be induced from those of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$, i.e. each family of surfaces can be obtained as quasilinear section of some ambient weighted $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ variety or some projective cone over it [6].

Theorem 1.3. *There exist at least seven biregular weight models of wellformed and quasismooth rigid del Pezzo surfaces of Fano index 1 or 2 such that their images under the anti-canonical or sub anti-canonical embeddings in* $\mathbb{P}^6(a,\dots,g)$ *are modeled on those of the Segre embedding of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *in* $\mathbb{P}^7$*. At least, five of them do not deform under* $\mathbb{Q}$*-Gorenstein degeneration to a toric variety.*

Theorem 1.4. *There exist at least five biregular index models of wellformed and quasismooth rigid del Pezzo surfaces in codimension 4; such that their images under their sub anti-canonical embeddings in* $\mathbb{P}^6(a,\dots,g)$ *can be described in terms of the equations of the Segre embedding of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *in* $\mathbb{P}^7$*.*

2 Computational steps of proofs

We explain the main tools used in the construction of models appearing in above theorems. The first key point of our computation is to prove the general Hilbert series formula for general weighted $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ variety w$\mathcal{P}$, which enables to compute the anti-canonical divisor class of the ambient variety and it's complete intersections. Then we use an algorithmic method [7,3] to search for candidate rigid del Pezzo surfaces by using the computer algebra system MAGMA [2]. We analyze these candidates to spot any pattern emerging among them which may lead a biregular model (infinite series) of surface. Once such a model is identified, we prove the existence of wellformed and quasismooth models.

2.1 Computer search algorithm

We recall an algorithm from [7] which we used to compute the list of candidate rigid del Pezzo surfaces. An important ingredient is the orbifold Riemann–Roch formula [4] which gives a formula for Hilbert series $P_X(t)$ of X as sum of a smooth part and an orbifold part. It states that if an orbifold X has a collection $\mathcal{B} = \{k_i \times Q_i : m_i \in \mathbb{Z}_{>0}\}$ of isolated orbifold points then;

$$P_X(t) = P_{\text{smooth}}(t) + \sum_{Q_i \in \mathcal{B}} k_i P_{Q_i}(t), \tag{2.1}$$

where $\mathbb{P}_{\text{smooth}}$ gives the smooth part and the represents the orbifold part of the Hilbert Series. Indeed if the variety is smooth then $P_X(t) = P_{\text{smooth}}(t)$. The algorithm provide a complete list of orbifolds having fixed Fano index, dimension and canonical class $K_X = \mathcal{O}(kD)$ in a given ambient space. Indeed, if X is a del Pezzo surface of index I then $k = -I$. We briefly recall its steps specifically of out algorithm for the case of del Pezzo surfaces.

(i) Calculate the Hilbert series and the canonical divisor class of the of w$\mathcal{P}$.
(ii) Find all possible embeddings in $\mathbb{P}^6(w_i)$ of index I, i.e. $K_X = \mathcal{O}(-I)$ by enumerating weights and using the adjunction formula.
(iii) For each embedding computed at step (ii), calculate the Hilbert series and the term representing its smooth part $P_{\text{smooth}}(t)$.
(iv) Compute the list of all possible rigid orbifold points from the weights of the $\mathbb{P}^6(w_i)$ containing X.
(v) Form all possible subsets of orbifold points and for each set $\mathcal{B}$, determine if the difference $P_X(t) - P_{\text{smooth}}(t)$ can be equal to $\sum_{Q_i \in \mathcal{B}} k_i P_{Q_i}(t)$ for some k_i.
(vi) X is a candidate rigid del Pezzo surface with a basket of singularities $\mathcal{B}$ if the multiplicities k_i are non-negative. Repeat step (iii) to (vi) for each embedding computed at step (ii).

2.2 Pattern analysis and existence

In this step, we analyze the data of candidate examples obtained by using a computer algorithm to spot a pattern among the candidates to form a model of del Pezzo surfaces. The key indicator of the pattern is usually the number of orbifold points and their multiplicities in the basket of orbifold point $\mathcal{B}$. Then we prove the existence of such models as a complete intersection of some w$\mathcal{P}$ or cone over it as a wellformed and quasismooth families of algebraic varieties.

The existence of a family of del Pezzo surfaces with the given Hilbert series follows straightforwardly from the existence of ambient w$\mathcal{P}$, though it does not guarantee that given del Pezzo surface has exactly the same singularities as those predicted by the computer search algorithm of 2.1 or it is quasismooth. The crucial part of the existence is to show that del Pezzo orbifold X contains exactly same orbifold points as suggested by the output in the computer search. The wellformedness can be deduced as a consequence of the existence with correct orbifold points, as X must avoid any orbifold locus of $\mathbb{P}^6(w_i)$ having dimension greater than or equal to one.

2.3 Quasismoothness and computer algebra

The proof of the quasismoothness requires an analysis of two different types of loci: orbifold loci and base loci. The orbifold loci are restrictions of the singular strata of $\mathbb{P}^6(w_i)$ to X and the base loci appear as loci of the linear systems of the successive intersecting weighted homogeneous forms while taking intersection with $\mathrm{w}\mathcal{P}$. A general member in each family del Pezzo surface appearing in these models is quasismooth, outside of these loci, due to Bertini's theorem. In practice, we show that the base loci remains constant in families for each $n \in \mathbb{Z}_+$ and prove the quasismoothness by using the computer algebra system Magma.

2.4 Non existence of degeneration to a toric variety

The first plurigenus $h^0(-K_X)$ (and all plurigenera $h^0(-mK)$) are invariant under $\mathbb{Q}$-Gorenstein deformations. Moreover $h^0(-K_X) > 0$ for any toric Fano variety, so each family with $h^0(-K_X) = 0$ does not admit a $\mathbb{Q}$-Gorenstein degeneration to a toric variety. In total, 5 out of 13 models have $h^0(-K_X) = 0$. These examples are of interest in a sense they can not be obtained using the standard toric techniques.

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