Investigation of the Dynamics of Two Connected Bodies in the Plane of a Circular Orbit Using Computer Algebra Methods

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Abstract. Computer algebra methods are used to investigate the equilibrium orientations of a system of two bodies connected by a spherical hinge that moves along a circular orbit under the action of gravitational torque in the plane of the orbit. An algebraic method based on the resultant approach is applied to reduce the satellite stationary motion system of algebraic equations to a single algebraic equation in one variable, which determines the equilibrium configurations of the two-body system in the orbital plane. Classification of domains with equal numbers of equilibrium solutions is carried out using algebraic methods for constructing discriminant hypersurfaces. Discriminant curves in the space of system parameters that determine boundaries of domains with a fixed number of equilibria for the two-body system are obtained symbolically. Using the proposed approach it is shown that the satellite–stabilizer system can have up to 24 equilibrium orientations in the plane of a circular orbit. Some simple cases of the problem were studied in detail.

Keywords: Two connected bodies · Satellite–stabilizer system · Spherical hinge · Gravitational torque · Circular orbit · Lagrange equations · Algebraic equations · Equilibrium orientation · Computer algebra · Resultant · Discriminant hypersurface.

1 Introduction

This paper presents the results of applying computer algebra methods to study the dynamics of a two-body system (satellite and stabilizer) connected by a spherical hinge, that moves in gravitational field on a circular orbit in the plane of the orbit. We continue our investigation of the dynamics of a system of two connected bodies started in the papers [1], [2], [3] and [4].

The dynamics of various schemes for satellite–stabilizer gravitational orientation systems was discussed in many papers, some review of them can be found in paper [3]. Since the problem is very complicated, in the previous works only the simplest cases were considered, when the spherical hinge is located at the intersection of the satellite and stabilizer principal central axis of inertia. The application of computer algebra makes it possible to find solutions for more complicated cases of this problem.

The equilibrium orientations of two bodies system were found in papers [1], [2], [3] and [5] in the special case, when the spherical hinge is positioned at the intersection of the principal central axes of inertia of the satellite and stabilizer.

In paper [1], some classes of spatial equilibrium orientations of the satellite–stabilizer system in the orbital reference frame were analyzed, using the combination of linear algebra methods and Gröbner basis construction method [6] for the general values of two bodies parameters. In [2] and [5], the equilibrium orientations of two bodies system were found in the orbital plane. In paper [3], we studied the spatial equilibrium orientations of the satellite-stabilizer system in the orbital coordinate frame for the certain combinations of values of inertial and geometrical characteristics of the connected bodies when equations of stationary motions of the system depend on a single parameter, using Gröbner basis construction method.

In [7], [8], the existence of equilibrium orientations for the two-body system was investigated in the cases where one of the principal axes of inertia of the first and second bodies coincides with the normal to the orbital plane, radius vector, or tangent to the orbit. To determine the equilibria of the two-body system, a system of 12 algebraic equations was decomposed into a number of subsystems. To solve each subsystem of algebraic equations, algorithm for constructing Gröbner bases was employed. Some of the indicated algebraic subsystems were solved in [7], the other subsystems were solved in [8].

In [4], we studied the spatial equilibrium orientations of the two bodies system in the orbital coordinate frame in the plane tangent to the orbit, in the case where the spherical hinge is positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer.

In the present paper, we study the new problem to find the equilibrium orientations of the two bodies system in the orbital coordinate frame, in the orbital plane, when as in the previous work the spherical hinge was positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer. Here we found more effective than in [2] method to transform the trigonometric equations of stationary motions of the system to the algebraic one.

The resultant calculation approach was applied to reduce the stationary motion system of two algebraic equations to a single algebraic equation in one variable that determines equilibrium configurations of the two-body system in the orbital plane. Classification of domains with equal numbers of equilibrium solutions is carried out using algebraic methods for constructing discriminant hypersurface. Some simple cases of positioning a spherical hinge are studied in detail.

2 Equations of Motion

To write the equations of motion of two bodies system, we introduce the following right-handed Cartesian coordinate systems: $OXYZ$ is the orbital coordinate system, the OZ axis is directed along the radius vector connecting the Earth center of mass C and the center of mass O of the two–body system, the OX axis is directed along the linear velocity vector of the center of mass O, and the OY axis coincides with the normal to the orbital plane. The axes of coordinate systems $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$, are directed along the principal central axes of inertia of the first and the second body, respectively (Fig. 1).

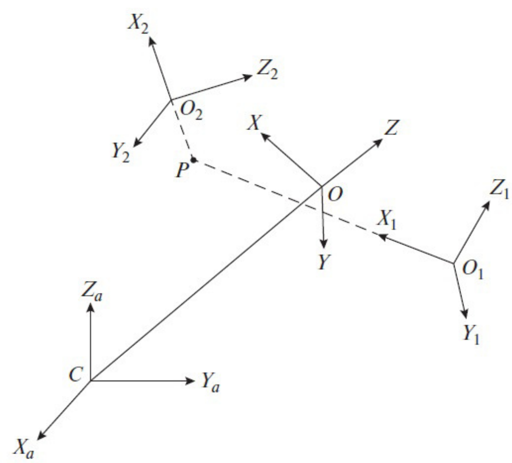

Fig. 1. Basic coordinate systems

The orientation of the coordinate system $O_ix_iy_iz_i$ with respect to the orbital coordinate system is determined by the aircraft angles α_i (pitch), β_i (yaw), and γ_i (roll) (see [3]).

Suppose that (a_i, b_i, c_i) are the coordinates of the spherical hinge P in the body coordinate system $Ox_iy_iz_i$, A_i, B_i, C_i are principal central moments of inertia; $M_1M_2/(M_1 + M_2) = M$; M_i is the mass of the ith body; ω_0 is the angular velocity for the center of mass of the two-body system moving along a circular orbit. Then we use the expressions for kinetic energy of the system in the case when $b_1 = b_2 = 0$ and the coordinates of the spherical hinge P in the body coordinate systems are $(a_i, 0, c_i)$ and when all the equilibrium configurations of the two-body system are located in the plane of the circular orbit

($\alpha_1 \neq 0, \alpha_2 \neq 0, \beta_1 = \beta_1 = 0$, $\gamma_1 = \gamma_2 = 0$) in the form [9]

$$
\begin{aligned}
T = {} & 1/2\big(B_1 + M(a_1^2 + c_1^2)\big)(\dot{\alpha_1} + \omega_0)^2 + 1/2\big(B_2 + M(a_2^2 + c_2^2)\big)(\dot{\alpha_2} + \omega_0)^2 \\
& - M\big((a_1a_2 + c_1c_2)\cos(\alpha_1 - \alpha_2) \\
& - (a_1c_2 - a_2c_1)\sin(\alpha_1 - \alpha_2)\big)(\dot{\alpha_1} + \omega_0)(\dot{\alpha_2} + \omega_0).
\end{aligned}
\tag{1}
$$

The force function, which determines the effect of the Earth gravitational field on the system of two connected by a hinge bodies, is given by [9]

$$
\begin{aligned}
U = {} & -3/2\omega_0^2\big((A_1 - C_1)\sin^2\alpha_1 + (A_2 - C_2)\sin^2\alpha_2\big) \\
& + 3/2M\omega_0^2\big((a_1\sin\alpha_1 - c_1\cos\alpha_1) - (a_2\sin\alpha_2 - c_2\cos\alpha_2)\big)^2 \\
& + M\omega_0^2\big((a_1a_2 + c_1c_2)\cos(\alpha_1 - \alpha_2) - (a_1c_2 - a_2c_1)\sin(\alpha_1 - \alpha_2)\big).
\end{aligned}
\tag{2}
$$

By using the kinetic energy expression (1) and the expression (2) for the force function, the equations of motion for this system can be written as Lagrange equations of the second kind by applying symbolic differentiation in the Wolfram Mathematica system [10], [11]

$$
\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha_i}} - \frac{\partial T}{\partial \alpha_i} - \frac{\partial U}{\partial \alpha_i} = 0, \quad i = \overline{1, 2}, \tag{3}
$$

in the form of a system of second-order ordinary differential equations in variables α_1 and α_2.

3 Equilibrium Orientations

Let us consider a stationary solution $(\alpha_1, \alpha_2) = (\alpha_1 = \text{const}, \alpha_2 = \text{const})$, also $A_i \neq C_i$. We obtain from (3) the stationary equations

$$
\begin{aligned}
&((A_1 - C_1)/M)\sin\alpha_1\cos\alpha_1 - (a_1\cos\alpha_1 + c_1\sin\alpha_1)(a_1\sin\alpha_1 - c_1\cos\alpha_1) \\
&\qquad = (a_1\cos\alpha_1 + c_1\sin\alpha_1)(c_2\cos\alpha_2 - a_2\sin\alpha_2), \\
&((A_2 - C_2)/M)\sin\alpha_2\cos\alpha_2 - (a_2\cos\alpha_2 + c_2\sin\alpha_2)(a_2\sin\alpha_2 - c_2\cos\alpha_2) \\
&\qquad = (a_2\cos\alpha_2 + c_2\sin\alpha_2)(c_1\cos\alpha_1 - a_1\sin\alpha_1),
\end{aligned}
\tag{4}
$$

which allow us to determine the equilibrium orientations of the system of two bodies connected by a spherical hinge in the orbital coordinate system. Equations (4) form a closed system of two equations with respect to the two aircraft angles α_1 and α_2, that determines the satellite–stabilizer equilibrium orientations in the orbital plane.

Trigonometric system (4) in the α_1 and α_2 angles cannot be solved directly. Therefore, for this system, we used the universal change of sines and cosines through the tangent

$$
\sin\alpha_i = \frac{\tan(\alpha_i)}{\sqrt{1 + \tan^2\alpha_i}} = \frac{t_i}{\sqrt{1 + t_i^2}}, \quad \cos\alpha_i = \frac{1}{\sqrt{1 + \tan^2\alpha_i}} = \frac{1}{\sqrt{1 + t_i^2}}, \tag{5}
$$

where $t_i = \tan \alpha_i$.

Substituting expressions (5) in terms of tangent into Eqs. (4) we obtain two equations with two unknowns t_1 and t_2. Then, the left-hand (right-hand) side of the first of these equations in terms t_1 and t_2 is first multiplied by the left-hand (right-hand) side of the second equation; then, the left-hand (right-hand) side of the first equation is divided by the left-hand (right-hand) side of the second equation, respectively. As a result, we obtain the algebraic system of two equations in two unknowns t_1 and t_2

$$
\begin{aligned}
f_0 t_1^3 + f_1 t_1^2 + f_2 t_1 + f_3 = 0, \\
h_0 t_1^2 + h_1 t_1 + h_2 = 0,
\end{aligned}
\tag{6}
$$

where f_i and h_i are the polynomials, which depend on t_2 and 6 system parameters a_i, c_i, and $d_i = (A_i - C_i)/M$.

Using the resultant concept we eliminate the variable t_1 from equations (6). Expanding the determinant of resultant matrix of Eqs.(6) with the help of Mathematica function [10], we obtain after factorization the 12th order algebraic equation in t_2 variable in the form

$$P(t_2) = P_1(t_2)P_2(t_2)P_3(t_2) = 0, \tag{7}$$

where $P_1(t_2)$ and $P_2(t_2)$ are the second degree polynomials and $P_3(t_2)$ is polynomial of the 8th degree. $P_1 = a_1 c_1 d_1 (a_2 t_2 - c_2)^2$ and $P_2(t_2) = a_2 c_2 t_2^2 + (a_2^2 - c_2^2 + d_2)t_2 + a_2 c_2$ and $P_3(t_2)$ has more complicated form.

By the definition of resultant, each root t_2 of Eq.(7) corresponds to one common root t_1 of system (6). The algebraic equation obtained has the even number of real roots, which does not exceed eight. We also have in (7) two quadratic equations $P_1(t_2) = 0$ and $P_2(t_2) = 0$. The total number of real roots, of these quadratic equations does not exceed four. By substituting real root of algebraic equation (7) into the second equations of system (6), we find the common root t_1 of these equations. It can be shown that two equilibrium solutions of the initial system (5) correspond to each real root t_2 of equation (7). Since the total number of real roots of (7) does not exceed 12, the satellite–stabilizer system in the orbital plane admits at most 24 equilibrium orientations in the orbital coordinate system. Using Eqs.(6) and (7) for each set of the system parameters, we can determine numerically the angles α_1 and α_2, that is all equilibrium orientations of the satellite–stabilizer system in the orbital coordinate system.

In a special case when $a_1 = a_2 = c_1 = c_2 = a$ we obtain from (7) a simpler algebraic equation $P_4(t_2) = 0$ of the 4th degree, whose coefficients depend only on two parameters d_{01} and d_{02} ($d_{01} = (A_1 - C_1))/Ma^2$, $d_{02} = (A_2 - C_2)/Ma^2$). To determine the conditions for the existence of real roots of equation $P_4(t_2) = 0$ we obtain an algebraic equation of the discriminant hypersurface. In our case, the discriminant hypersurface is given by the discriminant of polynomial $P_4(t_2)$. This hypersurface contains a component of codimension 1, which is the boundary of domains with an equal number of real roots. The set of singular points of the discriminant hypersurface in the plane of parameters d_{01} and d_{02} is given by the following system of algebraic equations: $P_4(t_2) = 0$ and $P_4'(t_2) = 0$. We eliminate the variable t_2 from the above system with the aid of the CAS Mathematica by using the call Resultant$[P_4(t_2), P_4'(t_2), t_2]$ and obtain an algebraic equation of the discriminant hypersurface whose coefficients depend only on two parameters d_{01} and d_{02}. Then using the algebraic equation of the discriminant hypersurface we determine the domains at the plane (d_{01}, d_{02}) with the equal number of equilibria. We used the results of the algebraic geometry for discriminants [12], [13], which yields that the domains with the same number of real zeroes of the polynomial are separated from each other by discriminant hypersurface.

We also consider the cases when one of the parameters a_1, a_2, c_1, or c_2 equals to zero. In these cases, for example, when $a_1 = 0$ we can obtain from (4) two equations: one quadratic and other of the 4th degree with two unknowns t_1 and t_2. The real roots of these equations determine the flat equilibrium orientations of the system of two bodies connected by a spherical hinge in the orbital coordinate system in these specials cases. We have investigated the conditions of the existence of real roots of these equations in detail.

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