Integrable Cases of the Polynomial Kind of the Liénard-type Equation*

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Abstract. The paper considers the relationship between the local integrability of an autonomous two-dimensional ODE system with polynomial right hand sides and its global integrability. A hypothesis is put forward that for the existence of the first integral of motion in a certain region of the phase space it needs the local integrability in the neighborhoods of all points of this domain.
Using the example of a polynomial case of the Liénard-type equation written as a dynamical system with parameters, we wrote out the conditions for local integrability near the stationary points and found the constraints on the parameters under which these conditions are satisfied. In this way we found several cases of integrability. Thus, we propose a heuristic method that allows one to determine the cases of integrability of an autonomous ODE with a polynomial right-hand side.

Keywords: Resonance normal form · Integrability · Liénard-type equation · Computer algebra.

Introduction

The article discusses a possible connection between the global and local integrability of two-dimension autonomous ODE systems. Global integrability, i.e. the existence of the first integral is an important property that allows to study the phase portrait of the system, construct symplectic integration schemes, and obtain a solution of the system in quadratures, i.e. the solvability of the system. The last is important, since the discovery of exactly solvable problems is one of the goals of the computer algebra.

Global integrability is a rather rare property, however, for ODE systems depending on parameters, one can set the task of finding such constraints on parameters at which the system is integrable.

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The main hypothesis is that in the domain of integrability in the phase space, the necessary condition is local integrability at every point of this domain. At all regular points the local integrability already takes place, so local integrability is necessary at singular points, including all stationary points.

We use an approach based on local analysis. It uses the resonance normal form computed near stationary points [1,2]. In the paper [3] this method was used for studying the degenerate two-dimensional autonomous ODE system.

Note that for the global integrability of an autonomous planar system, it suffices to have a single first integral of motion.

Problem

We will check the method on the example of the Liénard-like equation

$$\ddot{x} = f(x)\dot{x} + g(x), \tag{1}$$

Here we assume that $f(x)$ and $g(x)$ are polynomials. Usually, the Liénard equation assumes that $f(x)$ is an even function and $g(x)$ is an odd function [4] . We do not assume a certain parity for them, so we are talking about the Linard-type equation.

We parametrize equation (1) as the dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (a_0 + a_1 x)\, y + b_1 x + b_2 x^2 + b_3 x^3, \end{aligned} \tag{2}$$

here x and y are functions in time and parameters a_0, a_1, b_1, b_2, b_3 are real.

The problem is to obtain restrictions on the parameters that follow from the need for local integrability at stationary points, and then try to construct the corresponding first integrals of the system (2).

Local Analysis and the Resonance Normal Form

The local analysis applies for studying the property of the ODE system in the neighborhood of some points of the phase space. Near the regular points the system can be transformed to the linear system by an analytic transformation Ch.2 of [2], so, such points can not restrict the integrability. One should concentrate on singular points only. If we consider systems with polynomial right hand side (2), these are the stationary points. In their neighborhoods the resonance normal form is very effective for analysis of the local integrability.

The resonance normal form for nonresonant cases was introduced by A.Poincaré for the investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.

The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno et.al. This technique is based on the Local Analysis method by Prof. Bruno [1,2].

Condition of the Local Integrability

Let's suppose that we treat the n-dimension system with formal series in the right hand side

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q}\in\mathbb{N}^n} f_{i,\mathbf{q}} \mathbf{y}^{\mathbf{q}}, \quad i = 1, \ldots, n\,, \tag{3}$$

where we used the multi-index notation

$$\mathbf{y}^{\mathbf{q}} = \prod_{j=1}^{n} {y_j}^{q_j}\,,$$

with the power exponent vector $\mathbf{q} = (q_1, \ldots, q_n)$ and the sets

$$\mathcal{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,,\ \text{if } j \neq i\,, \quad j = 1, \ldots, n\}\,.$$

The i-th component of the vector $\mathbf{q}$ may be negative because the factor y_i has been moved out of the sum in (3).

Above we suppose that the origin is a stationary point and the linear part of the system have been reduced to the diagonal form by the linear transformation. λ_i are the eigenvalues of this linear part. Note, there are general formulas for the Jordan case of the linear part but we do not quote they here. The normalization is done with the near-identity transformation:

$$x_i = z_i + z_i \sum_{\mathbf{q}\in\mathbb{N}^n} h_{i,\mathbf{q}} \mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n \tag{4}$$

after which we have system (3) in the resonance normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\langle \mathbf{q}, \Lambda\rangle = 0} g_{i,\mathbf{q}} \mathbf{z}^{\mathbf{q}}, \quad i = 1, \ldots, n\,. \tag{5}$$

Transformation (4) keeps the linear part of the system.

The important difference between (3) and (5) is a restriction on the range of the summation, which is defined by the equation $\langle \mathbf{q}, \mathbf{\Lambda}\rangle = \sum_{j=1}^{n} q_j \lambda_j = 0$. Note, if the eigenvalues are not comparable then this condition is not valid for any components of the integer vector $\mathbf{q}$. So, if the ratio λ_1/λ_2 of the two-dimension system is not a not positive rational number then normal form (5) will be a linear system and, thus, integrable. In other words, for the limitation of the integrability, only the resonant cases have interest, when the ratio of the eigenvalues is rational and not positive.

Theorem 1 (Bruno [1]) *There exists a formal change (4) reducing the system (3) to its normal form (5)).*

Condition **A**. In the normal form (5) there exist formal power series $\alpha(\mathbf{z})$ and $\beta(\mathbf{z})$ such that

$$\sum_{\langle \mathbf{q}, \Lambda\rangle = 0} g_{i,\mathbf{q}} \mathbf{z}^{\mathbf{q}} = \lambda_i \alpha(\mathbf{z}) + \bar{\lambda}_i \beta \mathbf{z}), \quad i = 1, \ldots, n.$$

There is also the condition ω on λ_i. It is fulfilled for almost all eigenvalues. At least it is fulfilled in discussed case.

Theorem 2 (Bruno [1]) *If vector Λ satisfies the condition ω and the normal form (5) satisfies the condition* **A** *then the normalizing transformation (4) converges.*

The condition **A** is an infinite set of equalities. If the system under consideration depends on parameters then these set of equalities is the set of equations on the parameters.

Near a stationary point the condition **A** ensures convergence, provides the local integrability and isolates the periodic orbits if the eigenvalues are pure imagine. Here we look for the situation when this condition is satisfied.

Note, that if the infinite set of equation is the necessary condition then each subset is the necessary condition too. So we calculate the normal form up to some finite order and get the system of polynomial equations.

The algorithm for calculation of the normal form, and of the normalizing transformation with the corresponding computer program are described in [5].

First, we look for sets of parameters under which the condition **A** is satisfied at the stationary point of the system at the origin of (2). We solve the corresponding systems of algebraic equations with respect to the parameters a_0, a_1, b_1, b_2, b_3 and check the integrability at other stationary points for each found set of parameters. The received sets of parameters are good candidates for the existence of the first integral. These integrals are calculated by the method described in Appendix.

Resonant Case of the System

The normal form has a non-trivial form and brings some limitations in the resonant case only. This means that we can use our method if the ratio of eigenvalues of the linear part of the system (2) should be rational and not positive. The simplest (but not the only one) possibility to satisfy these conditions is to fix the parameter b_1 in system (2)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (a_0 + a_1 x)\, y + \tfrac{a_0^2 M}{(M-1)^2} x + b_2 x^2 + b_3 x^3, \quad M \neq 1. \end{aligned} \tag{6}$$

The matrix of the linear part of this system is

$$\begin{pmatrix} 0 & 1 \\ \frac{{a_0}^2 M}{(M-1)^2} & a_0 \end{pmatrix},$$

with eigenvalues

$$\left\{ \frac{-a_0}{M-1}, \frac{a_0 M}{M-1} \right\}.$$

Their ratio is -M. It corresponds to the resonance $1 : M$.

Conditions of the Local Integrability

We calculated the normal form for resonances (1:2), (1:3), and (1:4) till the terms of ninth, twelfth and fifteenth orders. For each case, we have obtained the algebraic systems in the parametric space with three equations for the each resonance. These three equations for $M = 2$ are

$$a_0[{a_0}^3 \left(2{a_1}^3 - 29a_1b_3\right) + {a_0}^2 b_2 \left(26{a_1}^2 + 43b_3\right) + 13a_0a_1{b_2}^2 - 11{b_2}^3] = 0,$$

$$\begin{aligned} &a_0[{a_0}^6 \left(6508{a_1}^6 - 56432{a_1}^4b_3 - 1187243{a_1}^2{b_3}^2 - 853200{b_3}^3\right) + \\ &2{a_0}^5a_1b_2 \left(45454{a_1}^4 + 524839{a_1}^2b_3 + 56431{b_3}^2\right) + \\ &{a_0}^4{b_2}^2 \left(39206{a_1}^4 + 1900239{a_1}^2b_3 + 2097973{b_3}^2\right) - 4{a_0}^3a_1{b_2}^3 \left(85609{a_1}^2 + 47372b_3\right) - \\ &{a_0}^2{b_2}^4 \left(349006{a_1}^2 + 932317b_3\right) + 43628a_0a_1{b_2}^5 + 95392{b_2}^6] = 0, \end{aligned}$$

$$\begin{aligned} &{a_0}^3[{a_0}^6(96473174{a_1}^9 + 61382001{a_1}^7b_3 - 25992267942{a_1}^5{b_3}^2 - 177095320637{a_1}^3{b_3}^3 - \\ &\quad 108865809000a_1{b_3}^4) + \\ &3{a_0}^5b_2(453029342{a_1}^8 + 9954001713{a_1}^6b_3 + 43938980394{a_1}^4{b_3}^2 - 82884194501{a_1}^2{b_3}^3 - \\ &\quad 94237137000{b_3}^4) + \\ &3{a_0}^4a_1{b_2}^2(58777497{a_1}^6 + 18754567420{a_1}^4b_3 + 185085944446{a_1}^2{b_3}^2 + \\ &\quad 149954547727{b_3}^3) + \\ &{a_0}^3{b_2}^3(-10471055755{a_1}^6 - 86771145246{a_1}^4b_3 + 281583701214{a_1}^2{b_3}^2 + \\ &\quad 445056171871{b_3}^3) + \\ &6a_0{b_2}^5 \left(1267606169{a_1}^4 - 6907374970{a_1}^2b_3 - 33195427246{b_3}^2\right) + \\ &7a_1{b_2}^6 \left(1748728465{a_1}^2 + 11316860901b_3\right)] = \\ &12{a_0}^5a_1{b_2}^4 \left(991433401{a_1}^4 + 17016366014{a_1}^2b_3 + 26519945973{b_3}^2\right) + \\ &3{a_0}^2{b_2}^7 \left(698320089{a_1}^2 - 9972352909b_3\right) + 5173440108a_0a_1{b_2}^8 + 955445018{b_2}^9. \end{aligned}$$

The all solutions of this algebraic system calculated by the MATHEMATICA-11 solver are

$$\begin{aligned} &a) \quad \{a_0 \to 0, b_2 \to 0\}, \\ &b) \quad \{b_2 \to -a_0a_1, b_3 \to 0\}, \\ &c) \quad \{b_2 \to -4a_0a_1/7, b_3 \to -6/49\,{a_1}^2\}, \\ &d) \quad \{b_2 \to -a_0a_1/3, b_3 \to -{a_1}^2/9\}, \\ &e) \quad \{b_2 \to 3a_0a_1, b_3 \to {a_1}^2\}, \\ &f) \quad \{a_1 \to 0, b_2 \to 0, b_3 \to 0\}, \end{aligned} \tag{7}$$

and the resonant condition is $b_1 \to 2a_0^2$ for $M = 2$. Note that adding equations from higher orders of the normal form cannot increase the number of solutions.

At all these sets of parameters we checked the integrability condition at the all other stationary points of (6) till the 21th order of the normal form.

Returning to writing systems of equations in the form of second-order equations, we obtain equations corresponding to the constraints (7)

$$
\begin{array}{ll}
a)^{**} & \ddot{x} = a_1 x\,\dot{x} + b_3 x^3, \\
b)^{*} & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - a_0 a_1 x^2, \\
c)^{**} & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - \frac{4}{7} a_0 a_1 x^2 - \frac{1}{49} 6 {a_1}^2 x^3, \\
d)^{*} & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, \\
e)^{*} & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x + 3 a_0 a_1 x^2 + {a_1}^2 x^3, \\
f)^{*} & \ddot{x} = a_0 \dot{x} + 2 a_0^2 x.
\end{array}
\qquad (8)
$$

The asterisk (*) marks the cases which were solved by the procedure DSolve of the MATHMATICA11 system and cases (**) were solved by the procedure dsolve of the MAPLE17. The solutions a), d) and e) were got in the implicit form. Results are in Appendix.

Solvable Equations from the Collection

We compared the equations (8) with the collection of the known integrable equations from section 2.2.3-2 of the book [6]:

1)$\ddot{y} = -\dot{y} - a\,y^3$,
-Duffing's equation with the resonance 1:1. It does not intersect with cases (8);

2)$\ddot{y} = -a\,y\,\dot{y} - b\,y^3 - c\,y$,
-a) is the partial case of 2) at $a_1 \to -a, b_3 \to -b, c \to 0$;

3)$\ddot{y} = (a\,y + 3\,b)\,\dot{y} - 2\,b^2\,y - a\,b\,y^2 + c\,y^3$,
- a) is the partial case of 3) at $a_1 \to a, b \to 0, b_3 \to c$;

4)$\ddot{y} = (3\,a\,y + b)\,\dot{y} + c\,y - a\,b\,y^2 - a^2 y^3$,
-d) is the partial case of 4) at $a_1 \to 3a, a_0 \to b, c \to 2a_0^2$,
-f) is the partial case of 4) at $a \to 0, a_0 \to b, c \to 2a_0^2$;

5)$\ddot{y} = \left(\frac{7}{2}\,a\,y + \frac{5}{2}\,b\right)\,\dot{y} - \frac{3}{2}\,b^2 y - c\,y^2 - \frac{3}{2}\,a^2\,y^3$,
- This equation does not intersect with cases (8).

Equations 1) and 5) above do not intersect with (8). Cases b), c) and e) are not reflected in that book, but are integrable. The case a) is also integrable.

Higher resonances

The same integrable cases take places for higher resonances too. For example, case d) has the form

$$
\begin{array}{lll}
d) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + 2{a_0}^2 x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, & \text{for } M = 2, \\
d) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \frac{3{a_0}^2}{4} x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, & \text{for } M = 3, \\
d) & \ddot{x} = (a_0 + a_1 x)\,\dot{x} + \frac{4{a_0}^2}{9} x - \frac{1}{3} a_0 a_1 x^2 - \frac{1}{9} {a_1}^2 x^3, & \text{for } M = 4.
\end{array}
$$

So, it is possible make a guess that the coefficient b_1 at x in right hand side is an arbitrary parameter c

$$\ddot{x} = (a_0 + a_1 x)\,\dot{x} + c\,x - \frac{1}{3}a_0 a_1 x^2 - \frac{1}{9}{a_1}^2 x^3.$$

If so, the case d) is exactly equation 4) at $a_1 \to 3a, a_0 \to b$.

Conclusions

On the basis of the hypothesis about the connection between global and local integrability, 6 integrable cases (8) of a Liénard-type equation with a polynomial right-hand side are algorithmically obtained. For all of them the integrability has been proved by a constructive method. For cases b), c) and e), we did not find statements about their integrability in the literature.

Note that a similar analysis has been already carried out for the degenerated two-dimensional autonomous ODE system [3].

Note also that the described approach to integrability studying is not limited to the two-dimensional case.

Appendix

We tried to calculate the first integrals above by this simple method. The autonomous system of the second order can be rewritten as the first order non-autunomous equation. Let

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y).$$

We divided the left and right sides of the system equations into each other. In result we have the first-order differential equations for $x(y)$ or $y(x)$

$$\frac{dx}{dy} = \frac{P(x,y)}{Q(x,y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

Then we tried to solve them by the MATHEMATICA-11 solver DSolve and for cases b) d) e) and f) got solutions $y(x)$ (or $x(y)$). Then we expressed the integration constant $C[1]$ as a function in $x, y(x)$ and replaced these variables with $x(t)$ and $y(t)$. Thus, we obtain integrals of motion. The resulting integrals has been verified by direct calculation of the time derivative along the system. For equation b) the first integral is

$$I_b(x(t),y(t)) = \frac{(a_1 x(t) - 2a_0)\sinh\left(\frac{1}{2}R(x(t),y(t))\right) + a_0 R(x(t),y(t))\cosh\left(\frac{1}{2}R(x(t),y(t))\right)}{(a_1 x(t) - 2a_0)\cosh\left(\frac{1}{2}R(x(t),y(t))\right) + a_0 R(x(t),y(t))\sinh\left(\frac{1}{2}R(x(t),y(t))\right)},$$

where

$$R(x(t), y(t)) = \sqrt{\frac{a_1(x(t)(a_1x(t) - 2a_0) - 2y(t))}{{a_0}^2}}.$$

For equation c) the first integral calculated by the MAPLE17 is

$$I_c(x(t), y(t)) = \sqrt{S(x(t), y(t))}(42a_1y(t)(2a_1x(t) - 7a_0)(7a_0 + a_1x(t)) - 2(7a_0 - 2a_1x(t))(7a_0 - 3a_1x(t))(7a_0 + a_1x(t))^2 - 147a_1^2y(t)^2))/ ((7a_0 + a_1x(t))\sqrt{6S(x(t), y(t)) - 28}((7a_0 + a_1x(t))(7a_0 - 3a_1x(t)) + 14a_1y(t)),$$

where

$$S(x(t), y(t)) = \frac{(7a_0 + a_1x(t))^2}{2a_0(7a_0 + a_1x(t)) + a_1y(t)}.$$

For the cases d) and e) we found first integrals in the implicit form

$$F(x(t), y(t), Int(x(t), y(t))) = 0.$$

For f) case the first integral is

$$I_f(x(t), y(t)) = (2a_0x(t) - y(t))^2(a_0x(t) + y(t)).$$

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References

1. A.D. Bruno, *Analytical form of differential equations (I, II)*. Trudy Moskov. Mat. Obsc. **25**, 119–262 (1971), **26**, 199–239 (1972) (in Russian) = Trans. Moscow Math. Soc. **25**, 131–288 (1971), **26**, 199–239 (1972) (in English)
2. A.D. Bruno, *Local Methods in Nonlinear Differential Equations*. Nauka, Moscow 1979 (in Russian) = Springer-Verlag, Berlin (1989) P.348.
3. A.D. Bruno, V.F. Edneral, V.G. Romanovski, *On new integrals of the Algaba-Gamero-Garcia system*. Proceedings of the CASC 2017, Springer-Verlag series: LNCS **10490** (2017) 40–50
4. A. Liénard, Etude des oscillations entretenues, Revue générale de l'électricité **23** (1928) 901–912 and 946–954.
5. V.F. Edneral, R. Khanin, Application of the resonant normal form to high-order nonlinear odes using mathematica, Nuclear Instruments and Methods in Physics Research, Section A: Accelerators, Spectrometers, Detectors and Associated Equipment, **502**, no. 2-3 (2003) 643–645.
6. A.D. Polyanin, V.F. Zaitsev, *Handbook of exact solutions for ordinary differential equations–2nd ed.* Chapman & Hall, CRC Press (2003) P. 803.