A review on computational aspects of polynomial amoebas *

Vitaly A. Krasikov[0000−0001−9686−8493]

Plekhanov Russian University of Economics, Moscow, Russia
krasikov.va@rea.ru

Abstract. We review results of papers written on the topic of polynomial amoebas with an emphasis on the computational aspects of the topic. The polynomial amoebas have a lot of applications in various domains of science. Computation of the amoeba for a given polynomial and describing its properties is in general a problem of formidable complexity. We describe the main algorithms for computing and depicting the amoebas and geometrical objects associated with them, such as contours and spines. We review the latest software packages for computing the polynomial amoebas and compare their functionality and performance.

Keywords: polynomial amoebas · Newton polytope · visualization algorithms.

1 Introduction

An amoeba (in the mathematical sense of this word) of a polynomial is the image of its zero locus under the logarithmic map. Initially this notion was introduced by I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky in 1994 [4]. The polynomial amoebas have numerous applications in topology, dynamical systems [12], complex analysis [3], mirror symmetry [11], measure theory [7,10]. There are a lot of surveys published on the polynomial amoebas (see [6,15,14]), but not so many of them on the computational aspects of amoebas. Nevertheless, the computation of amoebas and their properties is a crucial part of important problems such as the membership problem or depicting an amoeba.

2 Notation

Let $p(z)$ be a (Laurent) polynomial in n complex variables:

$$p(z_1, \ldots, z_n) = \sum_{\alpha \in A} c_\alpha z^\alpha = \sum_{\alpha \in A} c_{\alpha_1 \ldots \alpha_n} z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n},$$

where $A \subset \mathbb{Z}^n$ is a finite set.

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Definition 1. *The amoeba* $\mathcal{A}_p$ of a polynomial $p(z)$ is the image of its zero locus under the map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$:

$$\mathrm{Log} : (z_1, \ldots, z_n) \longmapsto (\ln|z_1|, \ldots, \ln|z_n|).$$

There is a connection between characteristics of the amoeba for any polynomial and its Newton polytope.

Definition 2. The convex hull in $\mathbb{R}^n$ of the set A is called *the Newton polytope of* $p(z)$, we denote it by $\mathcal{N}_p$.

The connected components of the amoeba complement ${}^c\mathcal{A}_p = \mathbb{R}^n\backslash\mathcal{A}_p$ are convex subsets in $\mathbb{R}^n$. They are in bijective correspondence with the different Laurent expansions (centered at the origin) of the rational function $1/p(z)$ [4].

Definition 3. *The compactified amoeba* $\bar{\mathcal{A}}_p$ of a polynomial $p(z)$ is the closure of the image of its zero locus under the map $\nu : (z_1, \ldots, z_n) \longmapsto \frac{\sum_{\alpha\in A}|z^\alpha|\cdot\alpha}{\sum_{\alpha\in A}|z^\alpha|}$.

Definition 4. *The contour* of the amoeba $\mathcal{A}_p$ is the set $\mathcal{C}_p$ of critical points of the logarithmic map Log restricted to the zero locus of the polynomial $p(x)$.

The contour is the closed real-analytic hypersurface in $\mathbb{R}^n$, the boundary $\partial\mathcal{A}_p$ is a subset of the contour $\mathcal{C}_p$ but is in general different from it.

Let $\mathcal{A}' \subset \mathbb{R}^n \cap \mathcal{N}_p$ be a set of vectors α such that ${}^c\mathcal{A}_p$ contains components of the order α and

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{\log^{-1}(u)} \log|\frac{f(z)}{z^\alpha}| \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \ldots z_n}, \quad \forall\alpha \in \mathcal{A}', \quad u \in E_\alpha.$$

Consider the function $S_p : \mathbb{R}^n \to \mathbb{R}$:

$$S_p(x) = \max_{\alpha\in\mathcal{A}'}\{< \alpha, x > + a_\alpha\}$$

Definition 5. The set $\{x \in \mathbb{R}^n, \text{ where } S_p(x) \text{ is nonsmooth}\}$ is called *the spine of* $\mathcal{A}_p$.

Example 1. Consider the polynomial $p_1(z_1, z_2) = 5z_1 + 15z_1^2 + 8z_1^3z_2 + 10z_2^2 + 10z_1^3z_2^2 + 8z_2^3 + 15z_1z_2^4 + 5z_1^2z_2^4 + 50z_1z_2^3 + 50z_1^2z_2$. The Newton polytope, the amoeba, its spine, and the compactified amoeba of p_1 are shown in Figure 1.

3 Computation of Polynomial Amoebas

The problem of giving a complete geometric or combinatorial description for the amoeba of a polynomial has a significant computational complexity.

One of the basic problems of amoeba computation is whether the given point belongs to the amoeba or, equivalently, if it belongs to a component of the amoeba complement (the membership problem). It is stated in [12, Corollary 2.7] that this problem can be solved in polynomial time.

In two- and three-dimensional cases one of the simplest ways of describing the structure of amoeba is depicting it. For visualizing an amoeba we have to give the definition of the carcass of the amoeba first.

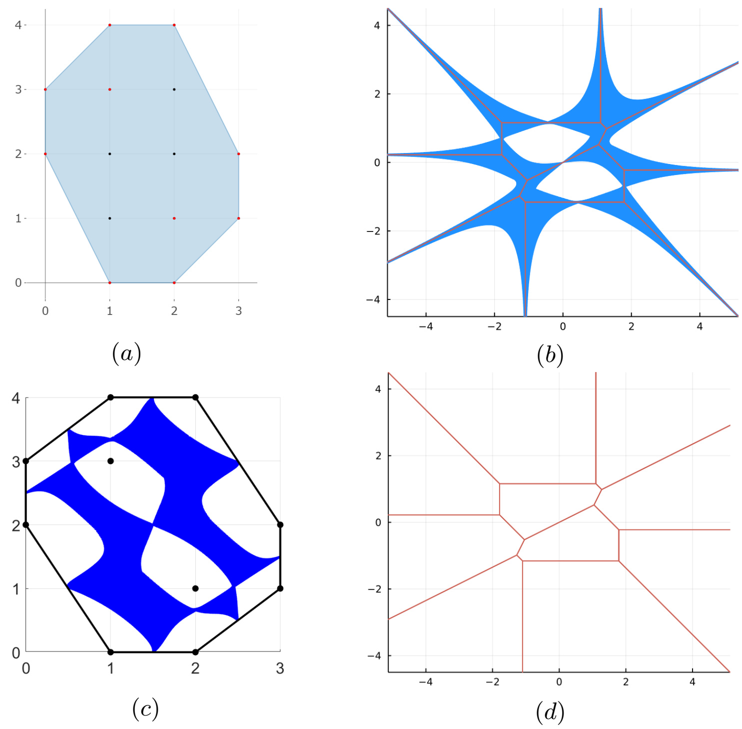

Fig. 1: The Newton polytope (a), the amoeba (b), the compactified amoeba (c) and the spine of the amoeba (d) of the polynomial p_1

Definition 6. *The carcass of the amoeba* $\mathcal{A}$ is any subset of $\mathcal{A}$ such that the number of connected components of the complement to the intersection $\mathcal{A} \cap B$ for a sufficiently large ball B is as big as it could possibly be (equal to the number of connected components in ${}^c\mathcal{A}$).

The carcass of an amoeba represents its significant properties (such as the number of connected components in its complement) and may be depicted by means of graphical packages. It follows from the definition that the carcass of any given amoeba is not unique. In what follows by depicting the amoeba we mean depicting its carcass.

3.1 Algorithms

The basic algorithm for depicting the amoeba of a bivariate polynomial is quite straightforward. It goes through the grid points in the logarithmic coordinates and checks the membership for the current point. Such algorithms in different variations are presented in [1,2,5,9]. Some of the improvements to the computing of amoebas include construction of the grid based on Archimedean tropical hypersurfaces and using greedy algorithms [13].

3.2 Software

Let us consider three of the newest software examples for depicting the amoebas and compare their functionality. All of these solutions are freeware and available online.

[a] The script by Dmitry Bogdanov at `http://dvbogdanov.ru/amoeba`. Very simple, both in usage and its functionality, it allows to depict amoebas and compactified amoebas for input polynomials. The script itself only generates the MatLab code for depicting the amoebas with given parameters. This approach has the disadvantage of needing a Matlab installation, but the benefit of being hardware-independent.

[b] Package PolynomialAmoebas written by Sascha Timme in the Julia programming language (`https://github.com/saschatimme/PolynomialAmoebas.jl`. It has very broad functionality, including not only tools for depicting the amoebas and not only in 2 dimensions. It also allows one to depict spines and contours of amoebas, coamoebas and amoebas in 3 dimensions.

[c] The project by Timur Sadykov and Timur Zhukov (`http://amoebas.ru/index.html`). It includes the visualization of amoebas in 2 and 3 dimensions, three-dimensional slices of 4D amoebas and allows to visualize the evolution of amoeba due to a change of the coefficients in the polynomial. The project is still under development.

3.3 Software comparison

In Table 1 the performance of the software tools mentioned above is compared. Seconds are chosen to be the units of measure there on purpose, since the time in milliseconds may vary slightly in different runs.

The packages differ from each other significantly, both in the implementation and the functionality, so we have to unify the parameters of functions somehow to compare their performance. A criterion in this case is the resemblance of the amoebas generated by the packages, so if this condition holds for some reasonable scale of the picture, we use the settings by default. Also changing of some parameters does not affect the computation time.

In the case of the script by D. Bogdanov important parameter is the number of points the algorithm depicts. It must be high enough for the picture to look solid, in Table 1 it equals 200. For Amoebas.ru one has to decide, how many iterations the algorithm performs and for the table this number is chosen to be 10.

Thorough analysis of the results of this comparison as well as the extended version of Table 1 will be available in the full version of this paper.

Table 1: The performance of the packages for computing the polynomial amoebas

Polynomial	[a], sec.	[b], sec.	[c], sec.
$1 + 8x + x^2 + 8y + 8xy + y^2$	4	2	6
$1 + 27x + 27x^2 + x^3 + 27y + 216xy + 27x^2y + 27xy^2 + y^3$	4	6	16
$1 + 125x + 1000x^2 + 1000x^3 + 125x^4 + x^5 + 125y + 8000xy + 27000x^2y + 8000x^3y + 125x^4y + 1000y^2 + 27000xy^2 + 27000x^2y^2 + 1000x^3y^2 + 1000y^3 + 8000xy^3 + 1000x^2y^3 + 125y^4 + 125xy^4 + y^5$	5	12	50
$1 + 625x + 10000x^2 + 10000x^3 + 625x^4 + x^5 + 625y + 160000xy + 810000x^2y + 160000x^3y + 625x^4y + 10000y^2 + 810000xy^2 + 810000x^2y^2 + 10000x^3y^2 + 10000y^3 + 160000xy^3 + 10000x^2y^3 + 625y^4 + 625xy^4 + y^5$	5	12	42
$1 + 144x + 4356x^2 + 48400x^3 + 245025x^4 + 627264x^5 + 853776x^6 + 627264x^7 + 245025x^8 + 48400x^9 + 4356x^{10} + 144x^{11} + x^{12} + 144y + 17424xy + 435600x^2y + 3920400x^3y + 15681600x^4y + 30735936x^5y + 30735936x^6y + 15681600x^7y + 3920400x^8y + 435600x^9y + 17424x^{10}y + \ldots + 435600x^2y^9 + 48400x^3y^9 + 4356y^{10} + 17424xy^{10} + 4356x^2y^{10} + 144y^{11} + 144xy^{11} + y^{12}$	10	-	207

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